Outline: Line Integrals

1. **Scalar Line Integrals**

   Let $\mathcal{C}$ be a curve with parameterization $\vec{x}(t) = \big(x(t), y(t)\big)$ for $a \le t \le b$, and let $f(x,y)$ be a scalar-valued function. Then the **integral of f along $\mathcal{C}$** is defined as follows:

$$\int_{\mathcal{C}} f(x,y)\, ds \;=\; \int_a^b f\big(\vec{x}(t)\big)\, \|\vec{x}\,'(t)\|\, dt \;=\; \int_a^b f\big(x(t), y(t)\big) \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

2. **Vector Fields and 1-Forms**

   A **vector field** is an assignment of a vector to every point on the plane. That is, a vector field is a function of the form

$$\vec{F}(x,y) \;=\; \big(P(x,y), Q(x,y)\big),$$

   where $P(x,y)$ and $Q(x,y)$ are scalar-valued functions. The corresponding **1-form** is the expression

$$P(x,y)\, dx \,+\, Q(x,y)\, dy.$$

3. **Vector Line Integrals**

   Let $\mathcal{C}$ be an oriented curve with parameterization $\vec{x}(t) = \big(x(t), y(t)\big)$ for $a \le t \le b$, and let $\vec{F}(x,y) = \big(P(x,y), Q(x,y)\big)$ be a vector field. Then the **integral of $\vec{F}$ along $\mathcal{C}$** is defined as follows:

$$\int_{\mathcal{C}} \vec{F}(x,y) \cdot d\vec{s} \;=\; \int_a^b \vec{F}\big(\vec{x}(t)\big) \cdot \vec{x}\,'(t)\, dt \;=\; \int_a^b \Big[P\big(x(t), y(t)\big)\, x'(t) \,+\, Q\big(x(t), y(t)\big)\, y'(t)\Big] dt$$

   Vector line integrals are sometimes written using 1-forms instead of vector fields:

$$\int_{\mathcal{C}} P(x,y)\, dx + Q(x,y)\, dy \qquad \text{instead of} \qquad \int_{\mathcal{C}} \vec{F}(x,y) \cdot d\vec{s}.$$

   The two notations mean exactly the same thing. In particular:

$$\int_{\mathcal{C}} P(x,y)\, dx + Q(x,y)\, dy \;=\; \int_a^b \Big[P\big(x(t), y(t)\big)\, x'(t) \,+\, Q\big(x(t), y(t)\big)\, y'(t)\Big] dt.$$

4. **The Fundamental Theorem of Calculus for Line Integrals**

   If $f(x,y)$ is a two-variable function, its **differential** is the 1-form

$$df \;=\; \frac{\partial f}{\partial x}(x,y)\, dx \,+\, \frac{\partial f}{\partial y}(x,y)\, dy.$$

The **fundamental theorem of calculus for line integrals** states that

$$\int_{\mathcal{C}} df \;=\; f(\vec{v}) - f(\vec{u})$$

where $\vec{u}$ is the begin point of the curve $\mathcal{C}$, and $\vec{v}$ is the end point of $\mathcal{C}$.

Using vector field notation, this theorem can be written

$$\int_{\mathcal{C}} \nabla f(x,y) \cdot d\vec{s} \;=\; f(\vec{v}) - f(\vec{u})$$

where $\nabla f(x,y) \;=\; \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$ is the **gradient** of f. In this form, this theorem is sometimes called the **gradient theorem**.

5. Green's Theorem

Let $\mathcal{C}$ be a simple, counterclockwise closed curve, let $\mathcal{R}$ be the region inside $\mathcal{C}$, and let $P(x,y)\,dx + Q(x,y)\,dy$ be a 1-form. Then **Green's Theorem** states that

$$\int_{\mathcal{C}} P(x,y)\,dx + Q(x,y)\,dy \;=\; \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right) dA$$

where the integral on the right is a double integral.

This theorem is especially useful for evaluating double integrals on the region inside a closed parametric curve. For example, the area inside a closed curve is given by the formula

$$\text{area} \;=\; \iint_{\mathcal{R}} dA \;=\; \int_{\mathcal{C}} -y\,dx \;=\; \int_{\mathcal{C}} x\,dy.$$

6. Conservative Vector Fields

A vector field $\vec{F}(x,y) = \big(P(x,y), Q(x,y)\big)$ is called **conservative** if

$$\frac{\partial P}{\partial y}(x,y) \;=\; \frac{\partial Q}{\partial x}(x,y)$$

for all x and y. In this case, the corresponding 1-form $P(x,y)\,dx + Q(x,y)\,dy$ is said to be **closed**.

Because mixed partial derivatives are equal, the gradient $\nabla f(x,y)$ of any function $f(x,y)$ is conservative. Conversely, if $\vec{F}(x,y)$ is a conservative vector field, there always exists a function $f(x,y)$ for which $\nabla f(x,y) = \vec{F}(x,y)$.

Similarly, the differential df of any function $f(x,y)$ is a closed 1-form, and any closed 1-form is the differential of some function.

Exercises: Line Integrals

1–3 ■ Evaluate the given scalar line integral.

1. $\int_{\mathcal{C}} y\,ds$, where $\mathcal{C}$ is the curve $\vec{x}(t) = (3\cos t, 3\sin t)$ for $0 \leq t \leq \pi/2$.

2. $\int_{\mathcal{C}} xy\,ds$, where $\mathcal{C}$ is the line segment between the points $(3,2)$ and $(6,6)$.

3. $\int_{\mathcal{C}} \left(x^2 + y^2\right) ds$, where $\mathcal{C}$ is the polar curve $r = e^{\theta}$ for $0 \leq \theta \leq \pi$.

4–6 ■ Evaluate the given vector line integral.

4. $\int_{\mathcal{C}} (y, 1) \cdot d\vec{s}$, where $\mathcal{C}$ is the curve $\vec{x}(t) = (t^3 - t, t^2)$ from the point $(0,0)$ to the point $(6,4)$.

5. $\int_{\mathcal{C}} xy\,dy$, where $\mathcal{C}$ is the portion of the ellipse $4x^2 + 9y^2 = 36$ lying in the first quadrant, oriented clockwise.

6. $\int_{\mathcal{C}} y\,dx - x\,dy$, where $\mathcal{C}$ is the portion of the curve $y = 1/x$ from the point $(1,1)$ to the point $(2, 1/2)$.

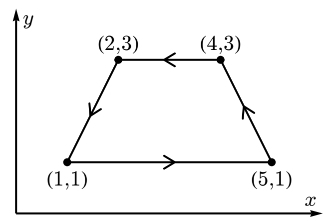
7–8 ■ Use the fundamental theorem of calculus for line integrals to evaluate the given integral.

7. $\int_{\mathcal{C}} ye^{xy}\,dx + xe^{xy}\,dy$, where $\mathcal{C}$ is a curve from the point $(0,0)$ to the point $(3,2)$.

8. $\int_{\mathcal{C}} (1 + \cosh y, x \sinh y) \cdot d\vec{s}$, where $\mathcal{C}$ is a curve from the point $(0,0)$ to the point $(1,1)$.

9–10 ■ Use Green's theorem to evaluate the given line integral.

9. $\int_{\mathcal{C}} x^{3/4} e^x\,dx + 3x\,dy$, where $\mathcal{C}$ is the closed curve shown in the following figure:



10. $\int_{\mathcal{C}} \left(\sqrt{x}\sin x, x^3 + 3xy^2\right) \cdot d\vec{s}$, where $\mathcal{C}$ is the circle $x^2 + y^2 = 1$, oriented counterclockwise.

11. Use Green's theorem to evaluate $\iint_{\mathcal{R}} y\,dA$, where $\mathcal{R}$ is the region inside the curve $\vec{x}(t) = \left(t - t^3, t^2\right)$ for $-1 \leq t \leq 1$.

12. Use Green's theorem to find the area of the region inside the curve $\vec{x}(t) = (\cos t, \sin t \cos t)$ for $-\pi/2 \leq t \leq \pi/2$.

13. Is the 1-form

$$y\cos(x^2y^2)\,dx \;+\; x\cos(x^2y^2)\,dy$$

the differential of some function? Explain.

Answers

1. 9 **2.** 95 **3.** $\dfrac{(e^{3\pi}-1)\sqrt{2}}{3}$ **4.** $\dfrac{308}{15}$ **5.** 4 **6.** $\log(4)$ **7.** e^6-1 **8.** $1+\cosh(1)$

9. 18 **10.** $\dfrac{3\pi}{2}$ **11.** $\dfrac{8}{35}$ **12.** $2/3$

13. Yes. It is a closed 1-form, since

$$\frac{\partial}{\partial y}\left[y\cos(x^2y^2)\right] \;=\; \cos(x^2y^2)-2x^2y^2\sin(x^2y^2) \;=\; \frac{\partial}{\partial x}\left[x\cos(x^2y^2)\right],$$

and hence it is the differential of some function $f(x,y)$.